

9/3/03

Ch. I Review of Banach, Hilbert and locally convex vector spaces

I. 0 Preliminaries & notations

Measure space (M, μ) (μ is σ -additive), $L^1(M, d\mu)$ - the set of integrable functions (complex-valued).

Three convergence theorems.

Thm 1 (The monotone convergence theorem)

If $f_n \in L^1(M, d\mu)$ and

$0 \leq f_1(x) \leq f_2(x) \leq \dots$ and

$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in M$, then

$f \in L^1(M, d\mu) \iff \lim_{n \rightarrow \infty} \|f_n\|_1 < \infty$.

In this case, $\lim_{n \rightarrow \infty} \|f - f_n\|_1 = 0$ and

$\lim_{n \rightarrow \infty} \|f_n\|_1 = \|f\|_1$.

(Here for $f \in L^1(M, d\mu)$,

$$\|f\|_1 = \int |f| d\mu < \infty)$$

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Thm 2 (The dominated convergence theorem) If $f_n \in L^1(M, d\mu)$,
 $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ a.e. on M and
if $\exists G \in L^1(M, d\mu)$ s.t.
 $|f_n(x)| \leq G(x)$ a.e. on $M \quad \forall n$,
then $f \in L^1(M, d\mu)$ and
 $\lim_{n \rightarrow \infty} \|f - f_n\|_1 = 0$.

Thm 3 (Fatou's lemma) If
 $f_n \in L^1(M, d\mu)$ and
 $f_n(x) \geq 0, \quad x \in M, \quad n \in \mathbb{N}$,
and if
 $\liminf_{n \rightarrow \infty} \|f_n\|_1 < \infty$,
then $f(x) = \liminf_{n \rightarrow \infty} f_n(x)$ is in
 $L^1(M, d\mu)$ and $\|f\|_1 \leq \liminf_{n \rightarrow \infty} \|f_n\|_1$.
Here $L^1(M, d\mu) = \mathcal{L}^1(M, d\mu)/\sim$
where $f \sim g$ if $f(x) = g(x)$ a.e.
(on M).

Def A normed linear space is a vector space V/\mathbb{C} (or \mathbb{R}) s.t.

$\exists \|\cdot\| : V \rightarrow \mathbb{R}$, satisfying

(i) $\|v\| \geq 0 \quad \forall v \in V$

(ii) $\|v\| = 0 \iff v = 0$

(iii) $\|\alpha v\| = |\alpha| \|v\|, \forall \alpha \in \mathbb{C}, v \in V$

(iv) $\|u+v\| \leq \|u\| + \|v\|, \forall u, v \in V$.

Def A normed linear space $(V, \|\cdot\|)$ is complete if it is complete as a metric space with the induced metric $d(u, v) = \|u-v\|$.

Def A Banach space (over \mathbb{C} or \mathbb{R}) is a complete normed space.

Thm (Riesz - Fisher)

$L^1(M, d\mu)$ is a Banach space.

Proof

Let $\{f_n\}$ be a Cauchy sequence. It is sufficient to prove that it contains a convergent subsequence (Verify!). Passing to a subsequence, we can assume that

$$\|f_n - f_{n+1}\|_1 \leq \frac{1}{2^n}.$$

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Let $g_m(x) = \sum_{n=1}^m |f_n(x) - f_{n+1}(x)|$ and

$g_\infty(x) = \lim_{m \rightarrow \infty} g_m(x)$. Since $g_m \uparrow g_\infty$ and $\lim_{m \rightarrow \infty} \|g_m\|_1 \leq \sum_{n=1}^{\infty} \|f_n - f_{n+1}\|_1 < 1$, by the MCT, $g_\infty \in L^1(M, \mu)$, so that

$$|g_\infty(x)| < \infty \text{ a.e. on } M.$$

For such x ,

$$f_m(x) = f_1(x) - \sum_{n=1}^{m-1} (f_n(x) - f_{n+1}(x)),$$

converges point-wise to $f(x)$. Since

$$|f_m(x)| \leq |f_1(x)| + |g_\infty(x)|$$

is in $L^1(M, \mu)$, then by the DCT,

$f \in L^1(M, \mu)$ and

$$\lim_{n \rightarrow \infty} \|f - f_n\|_1 = 0.$$

□

Example

$L^\infty(M, \mu) = \{ f : M \rightarrow \mathbb{C}, \text{ measurable,}$

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(or esup)

$$\|f\|_{\infty} = \text{vrai sup}_{x \in M} |f(x)|$$

$$= \inf_E \sup_{x \in M \setminus E} |f(x)|, \mu(E) = 0 \} / \sim$$

$L^{\infty}(M, \mu)$ is the Banach space
(Exercise)

This space ($M = \mathbb{C}$, μ - Lebesgue measure)
is very important in the theory of q.c.
mappings.

Def A vector space V is an inner
product space, if $\exists \mathbb{C}$

$(,) : V \times V \rightarrow \mathbb{C}$, satisfying

- (i) $(x, x) \geq 0 \wedge (x, x) = 0 \Leftrightarrow x = 0$
- (ii) $(x+y, z) = (x, z) + (y, z)$
- (iii) $(\alpha x, y) = \overline{\alpha} (x, y), \alpha \in \mathbb{C}$
- (iv) $(x, y) = \overline{(y, x)}$.

Lemma Every inner product space V
is a normed linear space with the
norm $\|x\| = (x, x)^{1/2} \geq 0$.

The proof is based on the Schwarz

Remark: Our definition of the inner product as linear w.r.t. the first argument is math. definition. In QM, physicists use Dirac bra and ket notation.

$$\langle x | y \rangle, |y\rangle \in \mathcal{H}$$

and it is linear with respect to the second argument. R-S use this convention.

inequality

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

(do the details!).

Proof of the Schwarz inequality :
suppose $\langle x, y \rangle \neq 0$ and set

$$\theta = \frac{\overline{\langle x, y \rangle}}{|\langle x, y \rangle|}, |\theta| = 1.$$

For all real t

$$\begin{aligned} 0 &\leq \|\theta x + ty\|^2 = (\theta x + ty, \theta x + ty) \\ &= \|x\|^2 + \theta t \langle x, y \rangle + \bar{\theta} t \langle y, x \rangle \\ &\quad + t^2 \|y\|^2 = \|y\|^2 t^2 + 2 |\langle x, y \rangle| t + \|x\|^2 \\ &\Rightarrow |\langle x, y \rangle|^2 - \|x\|^2 \|y\|^2 \leq 0. \end{aligned}$$

Moreover, equal sign means double root,
i.e. $\exists \lambda \in \mathbb{R}$ s.t.

$$\|\theta x + \lambda y\|^2 = 0, \text{ i.e.}$$

x & y are linear dependent.

Def A complete inner product space
is called a Hilbert space. An inner
product space is called a pre-Hilbert

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Remark This is a better proof
of Schwarz inequality than in R-S.

space.

Fact If V is a normed linear space, it can be completed: \exists Banach space \tilde{V} s.t. $V \hookrightarrow \tilde{V}$ isometrically as a dense subset.

I.1 Hilbert spaces

Def $\mathcal{H}_1 \simeq \mathcal{H}_2$ as Hilbert spaces, if $\exists U: \mathcal{H}_1 \rightarrow \mathcal{H}_2$, onto, satisfying

$$(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$$

$\forall x, y \in \mathcal{H}_1$.

Corollary $\ker U = \{0\}$

U is called isometry (unitary operator)

I.1.1 Geometry of Hilbert spaces

Def $x, y \in \mathcal{H}$ are orthogonal, $x \perp y$, if $(x, y) = 0$.

Thm (Pythagorean theorem) Let $\{x_n\}_{n=1}^N$ be an orthonormal set in

the pre-Hilbert space V . Then $\forall x \in V$,

$$\|x\|^2 = \sum_{n=1}^N |(x, x_n)|^2 + \left\| x - \sum_{n=1}^{N-1} (x, x_n) x_n \right\|^2$$

Proof (Exercise!)

Corollary (Bessel's inequality)

$$\|x\|^2 \geq \sum_{n=1}^N |(x, x_n)|^2$$

Corollary (the Schwarz inequality)

$$|(x, y)| \leq \|x\| \cdot \|y\|$$

Proof Use Bessel inequality with
 $N=1$ and $x_1 = \frac{y}{\|y\|}$ (provided $y \neq 0$)

I. 1. 2 Examples of Hilbert spaces

Ex. 1 (M, μ) - a measure space; set

$$\mathcal{H} = L^2(M, d\mu)$$

$= \left\{ f: M \rightarrow \mathbb{C}, \text{ measurable} : \right.$

$$\left. \|f\|^2 = \int_M |f|^2 d\mu \right\}$$

M

- Hilbert space with inner product

$$(f, g) = \int_M f \bar{g} d\mu$$

(Note that $f\bar{g} \in L^1(M, d\mu)$ since $|f(x)g(x)| \leq \frac{1}{2}(|f(x)|^2 + |g(x)|^2)$).

Finally, $L^2(M, d\mu)$ is complete. Indeed, let $\{f_n\}$ be a Cauchy sequence. Passing to a subsequence, we can assume that

$$\|f_n - f_{n+1}\| \leq \frac{1}{2^n} \text{ for all } n.$$

As before,

$$g_m(x) = \sum_{n=1}^m |f_n(x) - f_{n+1}(x)| \nearrow g_\infty(x)$$

and

$$\|g_\infty\|_{L^1}^{1/2} = \|g_m\| \leq \sum_{n=1}^m \|f_n - f_{n+1}\| \leq 1,$$

so by MCT,

$g_\infty \in L^2(M, d\mu)$ and $|g_\infty(x)| < \infty$
a.e. on M .

For such x ,

$$f_m(x) = f_1(x) - \sum_{n=1}^{m-1} (f_n(x) - f_{n+1}(x))$$

converges point-wise to $f(x)$. Since

$|f_m|^2(x) \leq |f_1(x)|^2 + |g_\infty(x)|^2$ is
in $L^1(M, d\mu)$, by DCT,

$$f^2 \in L^1(M, d\mu), \text{ i.e., } f \in L^2(M, d\mu)$$

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and $\|f\|^2 \leq \|f_1\|^2 + \|g_\infty\|^2$.

Repeating the same argument for the function $f - f_n$, we get

$$\|f - f_n\|^2 \leq \|g_\infty - g_{n-1}\|^2 \leq \frac{1}{4^{n-1}},$$

so that

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

Ex 2 $\mathcal{H} = \ell_2 = \left\{ \left(x_n \right)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ with the inner product $(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$.

Completeness of ℓ_2 - exercise

Ex 3 Let \mathcal{H} be a Hilbert space, (M, μ) be a measure space.

Def, $f : M \rightarrow \mathcal{H}$ is measurable if $\forall u \in \mathcal{H}$,

$$(f, u) : M \rightarrow \mathbb{C}$$

is measurable.

$$L^2(M, d\mu; \mathcal{H}) = \left\{ \text{measurable } f : M \rightarrow \mathcal{H} : \int_M \|f(x)\|_{\mathcal{H}}^2 d\mu(x) < \infty \right\}$$

$L^2(M, d\mu; \mathcal{H})$ is a Hilbert space
(exercise) - Hilbert space of vector
valued functions.

Ex 4 Direct sums.

$$\begin{aligned}\mathcal{H}_1 \oplus \mathcal{H}_2 &= \left\{ (x_1, x_2) : x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2 \right. \\ &\& \left((x_1, x_2), (y_1, y_2) \right) \\ &= (x_1, y_1) + (x_2, y_2)\end{aligned}$$

- Hilbert space. Similarly,

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ x = (x_n)_{n=1}^{\infty}, x_n \in \mathcal{H}_n \mid \right.$$
$$\left. \|x\|^2 = \sum_{n=1}^{\infty} \|x_n\|_{\mathcal{H}_n}^2 < \infty \right\}$$

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I.1.3 Orthogonal projection

$M \subset \mathcal{H}$ - closed subspace

Def $M^\perp = \{ x \in \mathcal{H} \mid x \perp y, \forall y \in M \}$

Thm 1 $\mathcal{H} = M \oplus M^\perp$.

(M^\perp is obviously closed!)

$\mathcal{B}_1, \mathcal{B}_2$ - Banach spaces,

$\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) = \{ T : \mathcal{B}_1 \rightarrow \mathcal{B}_2, \text{ linear \& bounded} \}$

(Note if T is linear, then the following is equivalent:

- (1) T is continuous at one point
- (2) T is —, — all points
- (3) T is bounded, $\|Tv\|_2 \leq C \|v\|_1$)

$\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a Banach space with respect to the norm

$$\|T\| = \sup_{\|v\|_1=1} \|Tv\|_2 \quad (\text{Exercise}).$$

In particular, $\mathcal{B}^* = \mathcal{L}(\mathcal{B}, \mathbb{C})$
- the dual space to \mathcal{B} (the space of continuous linear functionals on \mathcal{B}) is a Banach space.

Thm 2 (the Riesz lemma) $\mathcal{H}^* \cong \mathcal{H}$
- anti-isomorphism: $\forall \ell \in \mathcal{H}^* \exists! y \in \mathcal{H}$ s.t.

$$\ell(x) = (x, y) \quad \forall x \in \mathcal{H};$$

$$\|\ell\| = \|y\|.$$

Corollary A bounded sesquilinear form is a sesquilinear form of a bounded linear operator on \mathcal{H}

$$(B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C})$$

- (i) $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$
- (ii) $B(x, \alpha y + \beta z) = \bar{\alpha} B(x, y) + \bar{\beta} B(x, z)$

(iii) $|B(x, y)| \leq C \|x\| \|y\|, \forall x, y \in \mathcal{H}$.

Then $\exists! A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ s.t.

$$B(x, y) = (Ax, y), \forall x, y \in \mathcal{H}$$

Proof of Thm 1 is based on

Lemma Let $M \subset \mathcal{H}$ be a closed subspace and $x \in \mathcal{H}$. Then $\exists! z \in M$ s.t.

$$\|z - x\| \leq \|y - x\| \quad \forall y \in M.$$

Indeed, let $d = \inf \|y - x\|$. $\exists \{y_n\}_{y \in M}$

$\subset M$ s.t. $\|y_n - x\| \rightarrow d$.

Claim $\{y_n\}$ is a Cauchy sequence.

$$\begin{aligned} \|y_n - y_m\|^2 &= \|(y_n - x) - (y_m - x)\|^2 \\ &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|y_n + y_m - 2x\|^2 \\ (\text{since } \|a - b\|^2 + \|a + b\|^2 &= 2\|a\|^2 + 2\|b\|^2) \end{aligned}$$

$$\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4d^2 \rightarrow 0$$

as $m, n \rightarrow \infty$. Uniqueness - exercise.

Now proof of the Thm 1 (projection thm)

Let $x \in \mathcal{H}$; by lemma $\exists! z \in M$ closest to x . Set

$$w = x - z.$$

Claim $w \in M^\perp$.

Remark $\exists z_1, z_2 \in \mathbb{R}^2$ such points,

$$\|z_1 + z_2 - x\| \geq d,$$

$$\text{but } \|\frac{z_1 + z_2 - x}{2}\| \leq \frac{1}{2} \|z_1 - x\|.$$

$$+ \frac{1}{2} \|z_2 - x\| = d$$

\Rightarrow sign up triangle inequality

$$\Rightarrow z_1 - x = c(z_2 - x), \quad c \geq 0$$

and

$$d = \frac{1+c}{2} d, \quad (c \geq 1)$$

Proof Let $y \in M$, $(w, y) \neq 0$. Set $\lambda = \frac{1}{\|y\|^2}$

$$\tilde{z} = z + \frac{\lambda}{\|y\|^2} y \in M. \text{ Then}$$

$$\begin{aligned} \|\tilde{z} - x\|^2 &= (z + \frac{\lambda}{\|y\|^2} y - x, z - x + \frac{\lambda}{\|y\|^2} y) \\ &= \|z - x\|^2 - \frac{\lambda^2}{\|y\|^2} < \|z - x\|^2 \end{aligned}$$

- a contradiction! ($w = x - z$).

Proof of the Riesz lemma.

$M = \ker l \subset H$ - closed subspace;

$M = H \Rightarrow l = 0$. Otherwise, $\exists x_0 \in M^\perp$, $x_0 \neq 0$. Set

$$y = \frac{\overline{l(x_0)}}{\|x_0\|} x_0$$

If $x \in M$, $l(x) = (x, y) = 0$;

if $x = \alpha x_0$, $l(x) = \alpha (x_0, y) = \alpha l(x_0)$.

Finally, for $\forall x \in H$,

$$x = \left(x - \frac{\overline{l(x)}}{\overline{l(x_0)}} x_0 \right) + \frac{\overline{l(x)}}{\overline{l(x_0)}} x_0,$$

i.e.

$$H = M \oplus \mathbb{C} \cdot x_0.$$

$$\|\ell\| = \sup_{\|x\|=1} |(x, y)| \leq \|y\|,$$

and

$$\|\ell\| \geq \ell\left(\frac{y}{\|y\|}\right) = \|y\|.$$

Conversely, $\forall y \in \mathcal{H} \rightarrow ly \in \mathcal{H}^*$,
 $ly(x) = (x, y).$

Corollary $\mathcal{H}^{**} \cong \mathcal{H}.$

I. 1.4 Orthonormal bases

$S \subseteq \mathcal{H}$ - the maximal orthonormal set,
is called orthonormal basis.

Lemma 1 Orthonormal bases exist.

Proof Use Zorn's lemma.

Lemma 2 Let $S = \{x_\alpha\}_{\alpha \in A}$ be an
orthonormal basis in \mathcal{H} . Then $\forall x \in \mathcal{H}$,

$$x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha,$$

$$\|x\|^2 = \sum_{\alpha \in A} |(x, x_\alpha)|^2.$$

Conversely, if $\sum_{\alpha \in A} |c_\alpha|^2 < \infty$, then
 $\sum_{\alpha \in A} c_\alpha x_\alpha \in \mathcal{H}.$

Proof Use Bessel's inequality, which follows from Pythagorean theorem (for any finite orthonormal set) and completeness of \mathcal{H} .

Gram-Schmidt orthogonalization leads to the following result.

Lemma 3 A Hilbert space \mathcal{H} is separable (i.e., has a countable dense subset) iff it has a countable orthonormal basis S . If $\#(S) = N$, $\mathcal{H} \approx \mathbb{C}^N$, if $\#(S) = \infty$, $\mathcal{H} \approx \ell_2$ (non-canonically!).

Examples

• $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n \in \mathbb{Z}}$ is an orthonormal

basis in $L^2([0, 2\pi])$.

Proof (Different from R-S).

Let $f \in L^2(0, 2\pi)$ be s.t.

$$\int_0^{2\pi} f(t) e^{int} dt = 0 \quad \forall n \in \mathbb{Z}.$$

Set $F(t) = \int_0^t f(s) ds$, F -continuous
on $[0, 2\pi]$ and $(F(2\pi) = F(0) = 0)$

$$\int_0^{2\pi} F(t) e^{int} dt = 0 \quad \forall n \neq 0; \text{ then}$$

$$\text{for } \Phi(t) = F(t) - \int_0^{2\pi} F(s) ds$$

we have

$$\int_0^{2\pi} \Phi(x) e^{inx} dx = 0 \quad \forall n \in \mathbb{Z}.$$

By Weierstrass theorem, $\forall \varepsilon > 0 \exists$

$$\sigma(x) = \sum_{n=-N}^N a_n e^{inx}$$

s.t.

$$|\Phi(x) - \sigma(x)| < \varepsilon \quad \forall x \in [0, 2\pi].$$

Then

$$\|\Phi\|^2 = \int_0^{2\pi} |\Phi(x)|^2 dx$$

$$= \int_0^{2\pi} \Phi(x) \overline{(\Phi(x) - \sigma(x))} dx \leq \varepsilon \int_0^{2\pi} |\Phi(x)| dx \\ \leq \sqrt{2\pi} \|\Phi\| \varepsilon, \text{ i.e.}$$

$$\|\Phi\| \leq \varepsilon \sqrt{2\pi} \Rightarrow \|\Phi\| = 0.$$

I. 1.5 Tensor products

$\mathcal{H}_1 \otimes \mathcal{H}_2 =$ the completion of tensor product of \mathcal{H}_1 and \mathcal{H}_2 as vector spaces with respect to the inner product

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)_{\mathcal{H}_1} \cdot (x_2, y_2)_{\mathcal{H}_2}.$$

Lemma 1 If $\{\varphi_i\}$ and $\{\psi_j\}$ are orthonormal bases for \mathcal{H}_1 and \mathcal{H}_2 respectively, then $\{\varphi_i \otimes \psi_j\}$ is the orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Examples

- $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$
 $\approx L^2(M_1 \times M_2, d\mu_1 \times d\mu_2)$
- $L^2(M, d\mu) \otimes \mathcal{H} \approx L^2(M, d\mu; \mathcal{H})$

- Fock spaces. Set $\mathcal{H}^0 = \mathbb{C}$ and define

$$e^{\mathcal{H}} = "F(\mathcal{H})" = \bigoplus_{n=0}^{\infty} "\perp" \frac{1}{n!} \mathcal{H}^{\otimes n};$$

say for

$$\mathcal{H} = L^2(M, d\mu)$$

$$\Psi \in F(\mathcal{H}), \quad \Psi = (\varphi_0, \varphi_1(x_1), \varphi_2(x_1, x_2), \dots)$$

$$\|\Psi\|^2 = |\varphi_0|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_M |\varphi_n(x_1, \dots, x_n)|^2$$

$d\mu(x_1) \dots d\mu(x_n) < \infty.$

Set $S_n : \mathcal{H}^{\otimes^n} \rightarrow \mathcal{H}^{\otimes^n}$

by $S_n = \frac{1}{n!} \sum_{\sigma \in \text{Sym}(n)} \sigma,$

where

$$\sigma(x_1 \otimes \dots \otimes x_n) = x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)},$$

and

$$A_n = \frac{1}{n!} \sum_{\sigma \in \text{Sym}(n)} \varepsilon(\sigma) \sigma.$$

The

$$\mathcal{B}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^{\otimes^n}$$

and

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} A_n \mathcal{H}^{\otimes^n}$$

are boson and fermion Fock spaces.

I. 2 Banach spaces

I. 2.1 Examples • $L^p(M, d\mu)$, $p \geq 1$

$$\|f\|_p = \left(\int_M |f|^p d\mu \right)^{1/p} \quad 1 \leq p < \infty$$

The Minkowski inequality:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

The Hölder inequality

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad p, q, r \geq 1.$$

Then if $f \in L^p(M, d\mu)$, $g \in L^q(M, d\mu)$,
then $fg \in L^r(M, d\mu)$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

Lemma $L^p(M, d\mu)$ is complete (Riesz-Fisher)

• Sequence spaces $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$.

Def $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent on V , if $\exists C, C' > 0$ s.t.

$$C\|x\|_1 \leq \|x\|_2 \leq C'\|x\|_1 \quad \forall x \in V$$

I. 2.2 Duals

$\mathcal{B}^* = \mathcal{L}(\mathcal{B}, \mathbb{C})$ - the dual space of \mathcal{B} , the space of bounded linear functionals

Ex 1 $L^p(M, d\mu)^* \cong L^q(M, d\mu)$
~ complex anti-linear isometry, $1 < p < \infty$.
Thus $L^p(M, d\mu)$, $p > 1$, is reflexive:

$$L^p(M, d\mu)^{**} \cong L^p(M, d\mu)$$

(It is always $J: \mathcal{B} \hookrightarrow \mathcal{B}^{**}$,
an isometry).

Ex 2 $L^1(M, d\mu)^* \cong L^\infty(M, d\mu)$

(but $L^\infty(M, d\mu)^{**} \not\cong L^1(M, d\mu)$,
and $\nexists \mathcal{B}$ s.t. $\mathcal{B}^* = L^1(M, d\mu)$
- exercise)

I. 2.3 The Hahn-Banach theorem

Thm 1 Let V be a vector space/ \mathbb{R} and
 $p: V \rightarrow \mathbb{R}$ be convex function,

$p(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y),$
 $\alpha \in [0, 1]$, $x, y \in V$. Let $U \subset V$ be
a subspace and $\lambda: U \rightarrow \mathbb{R}$ linear
functional, satisfying $\lambda(x) \leq p(x), \forall x \in U$.
Then $\exists \Lambda: V \rightarrow \mathbb{R}$, linear, s.t.

$$\Lambda|_U = \lambda \text{ and } \Lambda(x) \leq p(x), x \in V.$$

The complex form of Hahn-Banach thm
is the following.

Thm Let V be a vector space / \mathbb{C} and $p: V \rightarrow \mathbb{R}$ satisfies

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y),$$

(absolutely convex)

$$x, y \in V, \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha| + |\beta| = 1.$$

Let $\lambda \in \text{Hom}(V, \mathbb{C})$, where $U \subset V$ is a subspace, $|\lambda(x)| \leq p(x)$, $x \in U$. Then $\exists \Lambda \in \text{Hom}(V, \mathbb{C})$ s.t.

$$\Lambda|_U = \lambda \text{ and } |\Lambda(x)| \leq p(x), x \in V.$$

Corollary 1 Let V be a normed linear space, $U \subset V$ a subspace, $\lambda \in U^*$ (continuous linear functionals on U).

Then $\exists \Lambda \in V^*$ s.t. $\Lambda|_U = \lambda$ & $\|\Lambda\|_{V^*} = \|\lambda\|_U$.

Proof Choose $p(x) = \|\lambda\|_U \|x\|$.

Corollary 2 Let $y \in V$. Then $\exists \Lambda \in V^*$, $\Lambda \neq 0$, s.t.

$$\Lambda(y) = \|\Lambda\|_{V^*} \|y\|.$$

Proof Use Corollary 1 with $U = \mathbb{C} \cdot y$ and $\lambda(cy) = c\|y\|$. (In fact, $\|\Lambda\|_{V^*} = 1$)

Corollary 3 Let $U \subset V$ a subspace, $y \in V$ s.t. $\text{dist}(y, U) = d$. Then

$\exists \Lambda \in V^*$ s.t. $\|\Lambda\| \leq 1$, $\Lambda(y) = d$,

$$\Lambda|_U = 0.$$

Hint for $L^\infty(M, d\mu)^* \neq L^1(M, d\mu)$:
Consider $U = C(M)$, $x_0 \in M$, and
define

$\lambda(f) = f(x_0)$, $f \in U$.
Extend it to $V = L^\infty(M, d\mu)$.

I. 2.4 Uniform boundedness and closed graph theorem.

Thm 1 (The principle of uniform boundedness)
Let \mathcal{B} be a Banach space, and
let

$$\mathcal{F} \subset \mathcal{L}(\mathcal{B}, V),$$

V - normed linear space. Suppose that
 $\forall x \in \mathcal{B}$ the set

$$\{ \|Tx\|, T \in \mathcal{F} \} \subset \mathbb{R}$$

is bounded. Then the set

$$\{ \|T\|, T \in \mathcal{F} \} \subset \mathbb{R}$$

is bounded.

Thm 2 (the open mapping theorem)

Let $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ ^{onto}. If

$U \subset \mathcal{B}_1$ is open, then $T(U) \subset \mathcal{B}_2$
is open.

Corollary (the inverse mapping theorem)

A continuous bijection $T: \mathcal{B}_1 \rightarrow \mathcal{B}_2$